

# Standing waves for coupled Schrödinger equations with decaying potentials\*

Zhijie Chen, Wenming Zou

*Department of Mathematical Sciences, Tsinghua University,  
Beijing 100084, China*

## Abstract

We study the following singularly perturbed problem for a coupled nonlinear Schrödinger system:

$$\begin{cases} -\varepsilon^2 \Delta u + a(x)u = \mu_1 u^3 + \beta uv^2, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta v + b(x)v = \mu_2 v^3 + \beta vu^2, & x \in \mathbb{R}^3, \\ u > 0, v > 0 \text{ in } \mathbb{R}^3, \\ u(x), v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{cases}$$

Here,  $a, b$  are nonnegative continuous potentials, and  $\mu_1, \mu_2 > 0$ . We consider the case where the coupling constant  $\beta > 0$  is relatively large. Then for sufficiently small  $\varepsilon > 0$ , we obtain positive solutions of this system which concentrate around local minima of the potentials as  $\varepsilon \rightarrow 0$ . The novelty is that the potentials  $a$  and  $b$  may vanish at someplace and decay to 0 at infinity.

## 1 Introduction

In this paper we consider standing wave solutions of time-dependent coupled nonlinear Schrödinger equations:

$$\begin{cases} -i\hbar \frac{\partial}{\partial t} \Phi_1 - \frac{\hbar^2}{2} \Delta \Phi_1 + a(x)\Phi_1 = \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1, & x \in \mathbb{R}^N, \ t > 0, \\ -i\hbar \frac{\partial}{\partial t} \Phi_2 - \frac{\hbar^2}{2} \Delta \Phi_2 + b(x)\Phi_2 = \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2, & x \in \mathbb{R}^N, \ t > 0, \\ \Phi_j = \Phi_j(x, t) \in \mathbb{C}, & j = 1, 2, \\ \Phi_j(x, t) \rightarrow 0, & \text{as } |x| \rightarrow +\infty, \ t > 0, \ j = 1, 2, \end{cases} \quad (1.1)$$

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\*Supported by NSFC (11025106). E-mail address: chenzhijie1987@sina.com(Chen); wzou@math.tsinghua.edu.cn (Zou)

where  $N \leq 3$ ,  $i$  is the imaginary unit,  $\hbar$  is the Plank constant,  $\mu_1, \mu_2 > 0$  and  $\beta \neq 0$  is a coupling constant. The system (1.1) appears in many physical problems, especially in nonlinear optics. Physically, the solution  $\Phi_j$  denotes the  $j^{th}$  component of the beam in Kerr-like photorefractive media (cf. [1]). The positive constant  $\mu_j$  is for self-focusing in the  $j^{th}$  component of the beam. The coupling constant  $\beta$  is the interaction between the two components of the beam. The problem (1.1) also arises in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states  $|1\rangle$  and  $|2\rangle$  (cf. [21]). Physically,  $\Phi_j$  are the corresponding condensate amplitudes,  $\mu_j$  and  $\beta$  are the intraspecies and interspecies scattering lengths. The sign of  $\beta$  determines whether the interactions of states  $|1\rangle$  and  $|2\rangle$  are repulsive or attractive, i.e., the interaction is attractive if  $\beta > 0$ , and the interaction is repulsive if  $\beta < 0$ , where the two states are in strong competition.

To obtain standing waves of the system (1.1), we set  $\Phi_1(x, t) = e^{-iEt/\hbar}u(x)$  and  $\Phi_2(x, t) = e^{-iEt/\hbar}v(x)$ . Then the system (1.1) is reduced to the following elliptic system

$$\begin{cases} -\frac{\hbar^2}{2}\Delta u + (a(x) - E)u = \mu_1 u^3 + \beta uv^2, & x \in \mathbb{R}^N, \\ -\frac{\hbar^2}{2}\Delta v + (b(x) - E)v = \mu_2 v^3 + \beta vu^2, & x \in \mathbb{R}^N, \\ u(x), v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{cases} \quad (1.2)$$

In this paper we are concerned with positive solutions for small  $\hbar > 0$ . For sufficiently small  $\hbar > 0$ , the standing waves are referred to as semiclassical states. Replacing  $a(x) - E, b(x) - E$  by  $a(x), b(x)$  for convenience, we turn to consider the following system

$$\begin{cases} -\varepsilon^2 \Delta u + a(x)u = \mu_1 u^3 + \beta uv^2, & x \in \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + b(x)v = \mu_2 v^3 + \beta vu^2, & x \in \mathbb{R}^N, \\ u > 0, v > 0 \text{ in } \mathbb{R}^N, \\ u(x), v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{cases} \quad (1.3)$$

where  $a, b$  are nonnegative continuous functions.

One of the difficulties in the study of (1.3) is that it has semi-trivial solutions of type  $(u, 0)$  or  $(0, v)$ . We call solutions  $(u, v)$  with  $u \not\equiv 0$  and  $v \not\equiv 0$  by nontrivial vector solutions (cf. [22]). A solution  $(u, v)$  with  $u > 0$  and  $v > 0$  is called a positive vector solution.

System (1.3) has been studied in Lin and Wei [24], Pomponio [31], Montefusco, Pellacci and Squassina [27] and Ikoma and Tanaka [22]. In [24], Lin and Wei studied (1.3) by analyzing least energy nontrivial vector solutions. When  $\beta > 0$ , they showed the existence of a least energy nontrivial vector solution for small  $\varepsilon > 0$  under suitable conditions on the behavior of  $a(x), b(x)$  as  $|x| \rightarrow +\infty$ . In [27], Montefusco, Pellacci and Squassina studied the case  $\beta > 0$ . They assume that  $a, b$  both have positive infimums and there exists  $z \in \mathbb{R}^N, r > 0$  satisfying

$$\min_{|x-z|<r} a(x) < \min_{|x-z|=r} a(x), \quad \min_{|x-z|<r} b(x) < \min_{|x-z|=r} b(x).$$

Then they showed for small  $\varepsilon > 0$  that (1.3) has a non-zero solution  $(u_\varepsilon, v_\varepsilon)$  such that  $u_\varepsilon + v_\varepsilon$  has exactly one global maximum point in  $\{x : |x - z| < r\}$ . However, when  $\beta > 0$  is small, one component of  $(u_\varepsilon, v_\varepsilon)$  converges to 0 (see Theorem 2.1 (ii) in [27]). In [22], Ikoma and Tanaka also considered the case  $\beta > 0$ . When  $\beta > 0$  is relatively small, they constructed a family of solutions of (1.3) which concentrates to a positive vector solution. We also refer to [24, 31] for the study of (1.3) when  $\beta < 0$ .

*Note that in all works [22, 24, 27, 31] mentioned above, they all assumed that  $a$  and  $b$  are positive bounded away from 0. In this paper, we consider the case where  $a, b$  may vanish at someplace and decay to 0 at infinity. In the sequel we assume that*

$$(\mathbf{V}_1) \quad a, b \in C(\mathbb{R}^N, \mathbb{R}) \text{ and } \inf_{x \in \mathbb{R}^N} a(x) \geq 0, \inf_{x \in \mathbb{R}^N} b(x) \geq 0.$$

$$(\mathbf{V}_2) \quad \liminf_{|x| \rightarrow +\infty} a(x)|x|^2 \log(|x|) > 0, \quad \liminf_{|x| \rightarrow +\infty} b(x)|x|^2 \log(|x|) > 0.$$

$$(\mathbf{V}_3) \quad \text{There exists a bounded open domain } \Lambda \text{ such that}$$

$$\inf_{x \in \bar{\Lambda}} a(x) = a_0 > 0, \quad \inf_{x \in \bar{\Lambda}} b(x) = b_0 > 0.$$

To study the concentration phenomena of solutions for system (1.3), the following constant coefficient problem plays an important role:

$$\begin{cases} -\Delta u + a(P)u = \mu_1 u^3 + \beta u v^2, & x \in \mathbb{R}^N, \\ -\Delta v + b(P)v = \mu_2 v^3 + \beta v u^2, & x \in \mathbb{R}^N, \\ u > 0, v > 0, & x \in \mathbb{R}^N, \\ u(x), v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{cases} \quad (1.4)$$

where  $P \in \bar{\Lambda}$ . Then  $a(P), b(P) > 0$  are positive constants. Note that system (1.4) appears as a limit problem after a suitable rescaling of (1.3). The existence and the asymptotic behavior of nontrivial vector solutions of (1.4) have received great interest recently, see [2, 3, 8, 11, 12, 16, 18, 23, 25, 26, 29, 30, 32, 33, 34] for example. Define  $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ . It is well known that solutions of (1.4) correspond to the critical points of  $C^2$  functional  $L_P : H \rightarrow \mathbb{R}$  given by

$$\begin{aligned} L_P(u, v) = & \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + a(P)u^2 + |\nabla v|^2 + b(P)v^2) dx \\ & - \frac{1}{4} \int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4) dx. \end{aligned} \quad (1.5)$$

Define the Nehari manifold

$$\mathcal{N}_P := \left\{ (u, v) \in H \setminus \{(0, 0)\}, \int_{\mathbb{R}^N} (|\nabla u|^2 + a(P)u^2 + |\nabla v|^2 + b(P)v^2) dx \right.$$

$$- \int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4) dx = 0 \}, \quad (1.6)$$

and a constant

$$\beta_0 := \max\{\mu_1, \mu_2\} \cdot \max_{x \in \bar{\Lambda}} \left\{ \frac{a(x)}{b(x)}, \frac{b(x)}{a(x)} \right\}. \quad (1.7)$$

By  $(V_1)$  and  $(V_3)$ , one has that  $0 < \beta_0 < \infty$ . With the help of [32, Theorem 2], we have the following

**Proposition 1.1.** *Let  $\beta > \beta_0$ . Then for any  $P \in \bar{\Lambda}$ , (1.4) has a positive radially symmetric vector solution  $(U_P, V_P) \in H$  which is a mountain-pass type solution and satisfies*

$$m(P) := L_P(U_P, V_P) = \inf_{(u,v) \in \mathcal{N}_P} L_P(u, v). \quad (1.8)$$

Moreover,  $P \mapsto m(P) : \bar{\Lambda} \rightarrow \mathbb{R}$  is continuous.

**Remark 1.1.** *We call a positive vector solution  $(U, V)$  satisfying (1.8) a positive least energy vector solution. So  $(U_P, V_P)$  is a positive least energy vector solution. By [10], we see that for  $\beta > 0$ , any positive solution of (1.4) is radially symmetric with respect to some point  $x_0 \in \mathbb{R}^N$ .*

By Proposition 1.1,  $m(P)$  is well defined and continuous in  $\bar{\Lambda}$ . Assume that

**(V<sub>4</sub>)** There exists a bounded smooth open domain  $O \subset \Lambda$  such that

$$m_0 := \inf_{P \in O} m(P) < \inf_{P \in \partial O} m(P).$$

**Remark 1.2.** *Assumption  $(V_4)$  is an abstract condition, since we can not write down explicitly the function  $m(P)$ . Such a type of abstract assumptions for system (1.4) can be seen in [22, 27]. This is also a general condition. In the special case of  $a(x) = b(x) + C$ , where  $C \geq 0$  is a constant, one can easily show that  $(V_4)$  holds if  $\inf_{P \in O} a(P) < \inf_{P \in \partial O} a(P)$ . More comments about assumption  $(V_4)$  can be seen in [22, Remarks 1.4-1.5].*

Define

$$\mathcal{M} := \{P \in O : m(P) = m_0\}. \quad (1.9)$$

Now we can state our main result.

**Theorem 1.1.** *Let  $N = 3$ ,  $\beta > \beta_0$  and assumptions  $(V_1) - (V_4)$  hold. Then there exists  $\varepsilon_0 > 0$ , such that for any  $\varepsilon \in (0, \varepsilon_0)$ , there exists a positive vector solution  $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$  of (1.3), which satisfies*

(i) *there exists a maximum point  $\tilde{x}_\varepsilon$  of  $\tilde{u}_\varepsilon + \tilde{v}_\varepsilon$  such that*

$$\lim_{\varepsilon \rightarrow 0} \text{dist}(\tilde{x}_\varepsilon, \mathcal{M}) = 0.$$

(ii) *for any such  $\tilde{x}_\varepsilon$ ,  $(w_{1,\varepsilon}(x), w_{2,\varepsilon}(x)) = (\tilde{u}_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon), \tilde{v}_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon))$  converge (up to a subsequence) to a positive least energy vector solution  $(w_1(x), w_2(x))$  of (1.4) with  $P = P_0$ , where  $\tilde{x}_\varepsilon \rightarrow P_0 \in \mathcal{M}$  as  $\varepsilon \rightarrow 0$ .*

(iii) For any  $\alpha > 0$ , there exists  $c, C > 0$  independent of  $\varepsilon > 0$  such that

$$(\tilde{u}_\varepsilon + \tilde{v}_\varepsilon)(x) \leq C \exp\left(-\frac{c}{\varepsilon} \frac{|x - \tilde{x}_\varepsilon|}{1 + |x - \tilde{x}_\varepsilon|}\right) (1 + |x - \tilde{x}_\varepsilon|)^{-1} |\log(2 + |x - \tilde{x}_\varepsilon|)|^{-\alpha}.$$

**Remark 1.3.** Since the potentials  $a, b$  satisfy  $(V_1) - (V_2)$ , in our proof of Theorem 1.1 we need to use the following Hardy inequality

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx \quad \forall u \in C_0^\infty(\mathbb{R}^N), \quad (1.10)$$

which holds for  $N \geq 3$ . That is why we assume  $N = 3$  in Theorem 1.1.

**Remark 1.4.** Assumption  $(V_2)$  implies that neither  $a$  nor  $b$  have compact supports. For the scalar case (see (1.11) below),  $(V_2)$  was introduced by Bae and Byeon [7]. Using Theorem 2 and Theorem 3 from Bae and Byeon [7], it is easily seen that, if

$$\limsup_{|x| \rightarrow +\infty} a(x)|x|^2 \log(|x|) = 0, \quad \limsup_{|x| \rightarrow +\infty} b(x)|x|^2 \log(|x|) = 0,$$

then system (1.3) has no nontrivial  $C^2$  solutions for any  $\varepsilon > 0$ . This is the reason that we assume  $(V_2)$  in Theorem 1.1.

For the scalar case

$$-\varepsilon^2 \Delta u + a(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad (1.11)$$

where  $1 < p < \frac{N+2}{N-2}$ , there are many works on the existence of solutions which concentrate and develop spike layers, peaks, around some points in  $\mathbb{R}^N$  while vanishing elsewhere as  $\varepsilon \rightarrow 0$ . For the case where  $\inf_{x \in \mathbb{R}^N} a(x) > 0$ , we refer to [13, 19, 20] and references therein. For the case where  $\inf_{x \in \mathbb{R}^N} a(x) = 0$ , we refer to [4, 5, 7, 14, 15, 28, 35] and references therein.

For such a type of system (1.3), as far as we know, there is no result on the case of potentials vanishing at someplace or decaying to 0 at infinity, and Theorem 1.1 seems to be the first result on this aspect. This paper is inspired by [7], however the method of their proof cannot work here because of our general assumption  $(V_4)$ . In fact, if we assume

$(V'_4)$  There is a bounded open domain  $O \subset \Lambda$  and  $x_0 \in O$  such that

$$a(x_0) = \inf_{x \in O} a(x) < \inf_{x \in \partial O} a(x), \quad b(x_0) = \inf_{x \in O} b(x) < \inf_{x \in \partial O} b(x),$$

instead of  $(V_4)$  in Theorem 1.1, then it seems possible to prove Theorem 1.1 by following Bae and Byeon' approach in [7]. It is easy to check that  $(V'_4)$  implies  $(V_4)$  but the inverse does not hold, so  $(V_4)$  is a more general assumption. Here we will prove Theorem 1.1 by developing further the methods in [7, 28]. Remark

that the approach in [28] cannot work directly in our paper, since, by their approach, one can only get the following decay estimate

$$\tilde{u}_\varepsilon(x) + \tilde{v}_\varepsilon(x) \leq C \exp\left(-\frac{c}{\varepsilon} \frac{|x - \tilde{x}_\varepsilon|}{1 + |x - \tilde{x}_\varepsilon|}\right) (1 + |x - \tilde{x}_\varepsilon|)^{-1},$$

which is not enough for us to show that  $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$  is a solution of (1.3) (because, in our following proof,  $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$  is obtained as a solution of a modified problem but not as a solution of the original problem (1.3)).

The rest of this paper proves Theorem 1.1, and we give some notations here. Throughout this paper, we denote the norm of  $L^p(\mathbb{R}^3)$  by  $|u|_p = (\int_{\mathbb{R}^3} |u|^p dx)^{\frac{1}{p}}$ , and the norm of  $H^1(\mathbb{R}^3)$  by  $\|u\| = \sqrt{|\nabla u|_2^2 + |u|_2^2}$ . We denote positive constants (possibly different) by  $C, c$ , and  $B(x, r) := \{y \in \mathbb{R}^N : |x - y| < r\}$ .

## 2 The constant coefficient problem

In this section, we study the constant coefficient problem (1.4) and prove Proposition 1.1. We assume  $N \leq 3$  here. First we recall a result from [32] about the following problem

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta u v^2, & x \in \mathbb{R}^N, \\ -\Delta v + \lambda v = \mu_2 v^3 + \beta v u^2, & x \in \mathbb{R}^N, \\ u > 0, v > 0, & x \in \mathbb{R}^N, \\ u(x), v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{cases} \quad (2.1)$$

where  $\mu_1, \mu_2, \lambda > 0$ . We denote  $L_P(u, v), \mathcal{N}_P$  by  $L(u, v), \mathcal{N}$  respectively when  $a(P), b(P)$  are replaced by 1,  $\lambda$  respectively. Then we have

**Theorem 2.1.** (see [32, Theorem 2(iv) and subsection 3.4]) Assume that  $\lambda \geq 1$ . Then for

$$\beta > \max\left\{\mu_1 \lambda, \mu_2 \lambda^{\frac{N}{2}-1}\right\},$$

problem (2.1) has a positive least energy vector solution  $(U, V) \in H$  which is a radially symmetric mountain-pass type solution and satisfies

$$L(U, V) = \inf_{(u, v) \in \mathcal{N}} L(u, v).$$

Moreover,

$$L(U, V) < \min\left\{\inf_{(u, 0) \in \mathcal{N}} L(u, 0), \inf_{(0, v) \in \mathcal{N}} L(0, v)\right\}.$$

Theorem 2.1 implies the following corollary immediately.

**Corollary 2.1.** Let  $\beta > \beta_0$ , where  $\beta_0$  is defined in (1.7). Then for any  $P \in \overline{\Lambda}$ , (1.4) has a positive least energy vector solution  $(U_P, V_P) \in H$  which is a radially symmetric mountain-pass type solution and satisfies (1.8) and

$$m(P) < \min\left\{\inf_{(u, 0) \in \mathcal{N}_P} L_P(u, 0), \inf_{(0, v) \in \mathcal{N}_P} L_P(0, v)\right\}. \quad (2.2)$$

Define

$$\begin{aligned}\mathcal{S}(P) &:= \{(u, v) \in H : L'_P(u, v) = 0, L_P(u, v) = m(P), \\ &\quad u > 0, v > 0, u, v \text{ are radially symmetric}\}, \\ \mathcal{S} &:= \{(P, u, v) \in \mathbb{R}^N \times H : P \in \overline{\Lambda}, (u, v) \in \mathcal{S}(P)\}.\end{aligned}\quad (2.3)$$

Let  $(u, v) \in H$  be any a nonnegative solution of (1.4) with  $L_P(u, v) = m(P)$ . Then (2.2) implies that  $u \not\equiv 0$  and  $v \not\equiv 0$ . Therefore we have  $u > 0$  and  $v > 0$  by the strong maximum principle. By Remark 1.1 there exists some  $x_0 \in \mathbb{R}^N$  such that  $(u(\cdot - x_0), v(\cdot - x_0)) \in \mathcal{S}(P)$ . We have the following properties.

**Lemma 2.1.** (i) *There exists  $C_0, C_1, C_2, C_3 > 0$  such that for all  $(P, u, v) \in \mathcal{S}$ , there hold*

$$\|u\|, \|v\| \leq C_0, \quad (2.4)$$

$$|u|_4, |v|_4 \geq C_1, \quad (2.5)$$

$$u(x), v(x), |\nabla u(x)|, |\nabla v(x)| \leq C_2 e^{-C_3|x|} \quad \forall x \in \mathbb{R}^N. \quad (2.6)$$

(ii)  $\mathcal{S}$  is compact in  $\mathbb{R}^N \times H$ .

(iii)  $m(P) : \overline{\Lambda} \rightarrow \mathbb{R}$  is continuous.

**Proof.** The proof is something standard. From (1.8) it is standard to see that

$$m(P) = \inf_{(u,v) \in \mathcal{N}_P} L_P(u, v) = \inf_{(u,v) \in H \setminus \{(0,0)\}} \max_{t>0} L_P(tu, tv). \quad (2.7)$$

(i) Let  $a_1 = \max_{x \in \overline{\Lambda}} a(x)$ . Then it is well known that

$$-\Delta u + a_1 u = \mu_1 u^3, \quad u \in H^1(\mathbb{R}^N)$$

has a positive solution  $U_0$  which is unique up to a translation. Then

$$\begin{aligned}\max_{t>0} L_P(tU_0, 0) &\leq \max_{t>0} \left( \frac{1}{2} t^2 \int_{\mathbb{R}^N} (|\nabla U_0|^2 + a_1 U_0^2) dx - \frac{1}{4} t^4 \int_{\mathbb{R}^N} \mu_1 U_0^4 dx \right) \\ &= \frac{1}{4} \int_{\mathbb{R}^N} (|\nabla U_0|^2 + a_1 U_0^2) dx.\end{aligned}$$

Combining this with (2.7) one has that  $m(P)$  is uniformly bounded for  $P \in \overline{\Lambda}$ . Since for any  $(P, u, v) \in \mathcal{S}$ ,

$$4m(P) = \int_{\mathbb{R}^N} (|\nabla u|^2 + a(P)u^2 + |\nabla v|^2 + b(P)v^2) dx,$$

we see from  $(V_3)$  that (2.4) holds. Recall that for any  $(P, u, v) \in \mathcal{S}$ ,  $u, v$  are radially symmetric, so we see from [9, Lemma A.II] that

$$u(x), v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty, \text{ uniformly for } (P, u, v) \in \mathcal{S}.$$

Using a comparison principle, we see that (2.6) holds. To prove (2.5), we assume by contradiction that there exists a sequence  $(P_n, u_n, v_n) \in \mathcal{S}$  such that

$$\lim_{n \rightarrow +\infty} |u_n|_4 = 0. \quad (2.8)$$

(The case  $|v_n|_4 \rightarrow 0$  is similar.) Passing to a subsequence,  $P_n \rightarrow P_0 \in \overline{\Lambda}$ . Define

$$H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\}.$$

Since  $L'_{P_n}(u_n, v_n) = 0$  and the Sobolev embedding  $H_r^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$  is compact, it is standard to show that  $(u_n, v_n)$  converges to some  $(u_0, v_0)$  strongly in  $H$  (up to a subsequence),  $L'_{P_0}(u_0, v_0) = 0$  and

$$\lim_{n \rightarrow \infty} m(P_n) = \lim_{n \rightarrow \infty} L_{P_n}(u_n, v_n) = L_{P_0}(u_0, v_0). \quad (2.9)$$

By (2.8), we get from  $L'_{P_n}(u_n, v_n)(u_n, 0) = 0$  that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + a(P_n)u_n^2) dx = \int_{\mathbb{R}^N} (\mu_1 u_n^4 + \beta u_n^2 v_n^2) dx \rightarrow 0, \text{ as } n \rightarrow \infty,$$

which implies that  $u_0 = 0$ . We denote  $L_P(u, v)$  by  $L_0(u, v)$  when  $a(P), b(P)$  are replaced by  $a_0, b_0$ . Then by a standard mountain-pass argument, there exists some  $\rho, \alpha > 0$  such that  $\inf_{\|u\|+\|v\|=\rho} L_0(u, v) = \alpha > 0$ . By (2.7) and  $(V_3)$  this means that

$$m(P) \geq \inf_{(u,v) \in H \setminus \{(0,0)\}} \max_{t>0} L_0(tu, tv) \geq \alpha > 0, \quad \forall P \in \overline{\Lambda}.$$

Therefore,  $v_0 \neq 0$ . By (2.2) we have  $m(P_0) < L_{P_0}(0, v_0)$ . On the other hand, let  $(U, V) \in \mathcal{S}(P_0)$ , then  $L_{P_0}(U, V) = m(P_0)$ . Note that

$$\begin{aligned} m(P_n) &\leq \max_{t>0} L_{P_n}(tU, tV) = \frac{(\int_{\mathbb{R}^N} (|\nabla U|^2 + a(P_n)U^2 + |\nabla V|^2 + b(P_n)V^2) dx)^2}{4 \int_{\mathbb{R}^N} (\mu_1 U^4 + 2\beta U^2 V^2 + \mu_2 V^4) dx} \\ &\rightarrow \frac{1}{4} \int_{\mathbb{R}^N} (|\nabla U|^2 + a(P_0)U^2 + |\nabla V|^2 + b(P_0)V^2) dx = m(P_0) \end{aligned}$$

as  $n \rightarrow \infty$ , that is,

$$\lim_{n \rightarrow \infty} m(P_n) \leq m(P_0) < L_{P_0}(0, v_0) = L_{P_0}(u_0, v_0), \quad (2.10)$$

a contradiction with (2.9). Hence, (2.5) holds.

(ii) For any sequence  $(P_n, u_n, v_n) \in \mathcal{S}$ , similarly as in the proof of (i), up to a subsequence, we may assume that  $P_n \rightarrow P_0$ ,  $(u_n, v_n) \rightarrow (u_0, v_0)$  strongly in  $H$  and  $(u_0, v_0)$  is a nontrivial vector solution of (1.3) with  $P = P_0$ . By (1.8) we have  $L_{P_0}(u_0, v_0) \geq m(P_0)$ . Meanwhile, (2.9) and (2.10) imply  $L_{P_0}(u_0, v_0) \leq m(P_0)$ . That is,  $L_{P_0}(u_0, v_0) = m(P_0) = \lim_{n \rightarrow \infty} m(P_n)$ . Since  $u_n, v_n > 0$  are radially symmetric, we also have that  $u_0, v_0 > 0$  are radially symmetric. Hence,  $(P_0, u_0, v_0) \in \mathcal{S}$ .

(iii) follows from the proof of (ii). This completes the proof.  $\square$

Proposition 1.1 follows directly from Corollary 2.1 and Lemma 2.1.  $\square$



### 3 Proof of Theorem 1.1

In this section we assume that  $N = 3$ ,  $\beta > \beta_0$  and assumptions  $(V_1) - (V_4)$  hold. Define  $a_\varepsilon(x) = a(\varepsilon x)$ ,  $b_\varepsilon(x) = b(\varepsilon x)$ . To study (1.3), it suffices to consider the following system

$$\begin{cases} -\Delta u + a_\varepsilon u = \mu_1 u^3 + \beta uv^2, & x \in \mathbb{R}^3, \\ -\Delta v + b_\varepsilon v = \mu_2 v^3 + \beta vu^2, & x \in \mathbb{R}^3, \\ u > 0, v > 0, & x \in \mathbb{R}^3, \\ u(x), v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{cases} \quad (3.1)$$

Let  $H_{a,\varepsilon}^1$  (resp.  $H_{b,\varepsilon}^1$ ) be the completion of  $C_0^\infty(\mathbb{R}^3)$  with respect to the norm

$$\|u\|_{a,\varepsilon} = \left( \int_{\mathbb{R}^3} |\nabla u|^2 + a_\varepsilon u^2 dx \right)^{\frac{1}{2}} \quad \left( \text{resp. } \|u\|_{b,\varepsilon} = \left( \int_{\mathbb{R}^3} |\nabla u|^2 + b_\varepsilon u^2 dx \right)^{\frac{1}{2}} \right).$$

Define  $H_\varepsilon := H_{a,\varepsilon}^1 \times H_{b,\varepsilon}^1$  with a norm  $\|(u, v)\|_\varepsilon = \sqrt{\|u\|_{a,\varepsilon}^2 + \|v\|_{b,\varepsilon}^2}$ .

From now on, for any set  $B \subset \mathbb{R}^3$  and  $\varepsilon, s > 0$ , we define  $B_\varepsilon := \{x \in \mathbb{R}^3 : \varepsilon x \in B\}$ ,  $B^s := \{x \in \mathbb{R}^3 : \text{dist}(x, B) \leq s\}$  and  $B_\varepsilon^s := (B^s)_\varepsilon$ . Without loss of generality, we may assume that  $0 \in \mathcal{M}$  and  $B(0, \rho_0) \subset O \subset B(0, \rho_1)$  for some  $\rho_1 > \rho_0 > 0$ . By  $(V_3) - (V_4)$  we can choose  $\delta \in (0, \rho_0)$  small such that  $\text{dist}(\mathcal{M}, \mathbb{R}^3 \setminus O) \geq 5\delta$  and

$$\inf_{x \in \overline{O^{5\delta}}} a(x) \geq a_0/2 > 0, \quad \inf_{x \in \overline{O^{5\delta}}} b(x) \geq b_0/2 > 0. \quad (3.2)$$

For  $0 < \varepsilon < \rho_0$  we define  $\gamma_\varepsilon : [\rho_0/\varepsilon, +\infty) \rightarrow (0, +\infty)$  by

$$\gamma_\varepsilon(t) := \frac{\varepsilon^2}{t^2 \log t}, \quad (3.3)$$

and

$$\chi_{O_\varepsilon}(x) := \begin{cases} 1 & \text{if } x \in O_\varepsilon, \\ 0 & \text{if } x \notin O_\varepsilon, \end{cases}$$

Denote  $F(s, t) := \frac{1}{4}(\mu_1 s^4 + 2\beta s^2 t^2 + \mu_2 t^4)$ , and set

$$F_\varepsilon(x, s, t) = \begin{cases} F(s, t) & \text{if } F(s, t) \leq \frac{1}{4}\gamma_\varepsilon^2(|x|), \\ \gamma_\varepsilon(|x|)\sqrt{F(s, t)} - \frac{1}{4}\gamma_\varepsilon^2(|x|) & \text{if } F(s, t) > \frac{1}{4}\gamma_\varepsilon^2(|x|). \end{cases} \quad (3.4)$$

Then we have

$$\nabla_{(s,t)} F_\varepsilon(x, s, t) = \begin{cases} (\mu_1 s^3 + \beta s t^2, \mu_2 t^3 + \beta s^2 t) & \text{if } F(s, t) \leq \frac{1}{4}\gamma_\varepsilon^2(|x|), \\ \gamma_\varepsilon(|x|) \frac{(\mu_1 s^3 + \beta s t^2, \mu_2 t^3 + \beta s^2 t)}{2\sqrt{F(s, t)}} & \text{if } F(s, t) > \frac{1}{4}\gamma_\varepsilon^2(|x|). \end{cases} \quad (3.5)$$

This means that  $F_\varepsilon(x, \cdot) \in C^1(\mathbb{R}^2)$  as a function of  $(s, t)$ . Define a truncated function

$$G_\varepsilon(x, s, t) := \chi_{O_\varepsilon}(x) F(s, t) + (1 - \chi_{O_\varepsilon}(x)) F_\varepsilon(x, s, t). \quad (3.6)$$

By the definition of  $\beta_0$  in (1.7), one has that  $\beta > \max\{\mu_1, \mu_2\}$ . Then it is easy to see that

$$G_\varepsilon(x, s, t) \leq F(s, t), \quad \forall x \in \mathbb{R}^3, \quad (3.7)$$

$$0 \leq 4G_\varepsilon(x, s, t) = \nabla_{(s,t)} G_\varepsilon(x, s, t)(s, t), \quad \forall x \in O_\varepsilon, \quad (3.8)$$

$$2G_\varepsilon(x, s, t) \leq \nabla_{(s,t)} G_\varepsilon(x, s, t)(s, t) \leq \sqrt{\beta} \gamma_\varepsilon(|x|)(s^2 + t^2), \quad \forall x \in \mathbb{R}^3 \setminus O_\varepsilon. \quad (3.9)$$

Define a functional  $J_\varepsilon : H_\varepsilon \rightarrow \mathbb{R}$  by

$$J_\varepsilon(u, v) := \frac{1}{2} \|u\|_{a,\varepsilon}^2 + \frac{1}{2} \|v\|_{b,\varepsilon}^2 - \int_{\mathbb{R}^3} G_\varepsilon(x, u^+, v^+) dx. \quad (3.10)$$

Here and in the following,  $u^+(x) := \max\{u(x), 0\}$  and so is  $v^+$ . By the following Hardy inequality in dimension  $N = 3$

$$\frac{1}{4} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} dx \leq \int_{\mathbb{R}^3} |\nabla u|^2 dx \quad \forall u \in C_0^\infty(\mathbb{R}^3), \quad (3.11)$$

it is standard to show that  $J_\varepsilon$  is well defined and  $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ . Furthermore, any critical points of  $J_\varepsilon$  are weak solutions of the following system

$$\begin{cases} -\Delta u + a_\varepsilon u = \partial_u G_\varepsilon(x, u^+, v^+), & x \in \mathbb{R}^3, \\ -\Delta v + b_\varepsilon v = \partial_v G_\varepsilon(x, u^+, v^+), & x \in \mathbb{R}^3, \\ u(x), v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{cases} \quad (3.12)$$

For each small  $\varepsilon > 0$ , we will find a nontrivial solution of (3.12) by applying mountain-pass argument to  $J_\varepsilon$ . Then we shall prove that this solution is a positive vector solution of (3.1) for  $\varepsilon > 0$  sufficiently small. This idea was first introduced by del Pino and Felmer [19].

**Lemma 3.1.** *Let  $\varepsilon \in (0, \varepsilon_1)$  be fixed, where  $\varepsilon_1$  satisfies*

$$\frac{\sqrt{\beta} \varepsilon_1^2}{\log \rho_0 / \varepsilon_1} = 1/8.$$

*For any  $c \in \mathbb{R}$ , let  $(u_n, v_n) \in H_\varepsilon$  be a  $(PS)_c$  sequence for  $J_\varepsilon$ , that is,*

$$J_\varepsilon(u_n, v_n) \rightarrow c, \quad J'_\varepsilon(u_n, v_n) \rightarrow 0.$$

*Then, up to a subsequence,  $(u_\varepsilon, v_\varepsilon)$  converge strongly in  $H_\varepsilon$ .*

**Proof.** Recall the definition of  $\gamma_\varepsilon$  in (3.3) and  $B(0, \rho_0) \subset O$ . By Hardy inequality (3.11), we have

$$\begin{aligned} \frac{1}{4} \int_{\mathbb{R}^3 \setminus O_\varepsilon} \sqrt{\beta} \gamma_\varepsilon(|x|)(u_n^2 + v_n^2) dx &\leq \frac{\sqrt{\beta} \varepsilon^2}{\log \rho_0 / \varepsilon} \frac{1}{4} \int_{\mathbb{R}^3} \frac{u_n^2 + v_n^2}{|x|^2} dx \\ &\leq \frac{\sqrt{\beta} \varepsilon^2}{\log \rho_0 / \varepsilon} \int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 dx \leq \frac{1}{8} \|(u_n, v_n)\|_\varepsilon^2. \end{aligned} \quad (3.13)$$

Therefore, we deduce from (3.8) and (3.9) that

$$\begin{aligned}
c + o(\|(u_n, v_n)\|_\varepsilon) &\geq J_\varepsilon(u_n, v_n) - \frac{1}{4} J'_\varepsilon(u_n, v_n)(u_n, v_n) \\
&= \frac{1}{4} \|(u_n, v_n)\|_\varepsilon^2 + \int_{\mathbb{R}^3} \left( \frac{1}{4} \nabla_{(u,v)} G_\varepsilon(x, u_n^+, v_n^+)(u_n, v_n) - G_\varepsilon(x, u_n^+, v_n^+) \right) dx \\
&\geq \frac{1}{4} \|(u_n, v_n)\|_\varepsilon^2 - \frac{1}{4} \int_{\mathbb{R}^3 \setminus O_\varepsilon} \sqrt{\beta} \gamma_\varepsilon(|x|) (u_n^2 + v_n^2) dx \\
&\geq \frac{1}{8} \|(u_n, v_n)\|_\varepsilon^2,
\end{aligned} \tag{3.14}$$

that is,  $\|(u_n, v_n)\|_\varepsilon \leq C$  for all  $n \in \mathbb{N}$ . Up to a subsequence, we may assume that  $(u_n, v_n) \rightharpoonup (u, v)$  weakly in  $H_\varepsilon$  and  $u_n \rightarrow u, v_n \rightarrow v$  strongly in  $L_{loc}^4(\mathbb{R}^3)$ . Since there exists  $\alpha_0 > 0$  dependent of  $\mu_1, \mu_2, \beta$  only, such that

$$\max\{\mu_1 s^2 + \beta t^2, \beta s^2 + \mu_2 t^2\} \leq \alpha_0 2\sqrt{F(s, t)}, \quad \forall s, t \in \mathbb{R}, \tag{3.15}$$

from (3.5) we obtain

$$|\partial_u G_\varepsilon(x, u^+, v^+)| \leq \alpha_0 \gamma_\varepsilon(|x|) |u|, \quad |\partial_v G_\varepsilon(x, u^+, v^+)| \leq \alpha_0 \gamma_\varepsilon(|x|) |v|, \quad \forall x \in \mathbb{R}^3 \setminus O_\varepsilon. \tag{3.16}$$

Then for any  $R \geq \rho_1$ , we deduce from (3.3), (3.11) and (3.16) that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \|u_n - u\|_{a, \varepsilon}^2 &= \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (\partial_u G_\varepsilon(x, u_n^+, v_n^+) - \partial_u G_\varepsilon(x, u^+, v^+))(u_n - u) dx \\
&\leq \limsup_{n \rightarrow \infty} \int_{B(0, R/\varepsilon)} (\partial_u G_\varepsilon(x, u_n^+, v_n^+) - \partial_u G_\varepsilon(x, u^+, v^+))(u_n - u) dx \\
&\quad + \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B(0, R/\varepsilon)} 2\alpha_0 \gamma_\varepsilon(|x|) (u_n^2 + u^2) dx \\
&\leq \limsup_{n \rightarrow \infty} C \left( \int_{B(0, R/\varepsilon)} |u_n - u|^4 dx \right)^{1/4} + C \frac{8\varepsilon^2 \alpha_0}{\log R/\varepsilon} = C \frac{8\varepsilon^2 \alpha_0}{\log R/\varepsilon}.
\end{aligned}$$

Since  $R \geq \rho_1$  is arbitrary, we see that  $u_n \rightarrow u$  strongly in  $H_{a, \varepsilon}^1$ . Similarly,  $v_n \rightarrow v$  strongly in  $H_{b, \varepsilon}^1$ . This completes the proof.  $\square$

For any  $\varepsilon \in (0, \varepsilon_1)$  fixed, we define

$$c_\varepsilon := \inf_{\gamma \in \Phi_\varepsilon} \sup_{t \in [0, 1]} J_\varepsilon(\gamma(t)), \tag{3.17}$$

where  $\Phi_\varepsilon = \{\gamma \in C([0, 1], H_\varepsilon) : \gamma(0) = (0, 0), J_\varepsilon(\gamma(1)) < 0\}$ .

**Lemma 3.2.** *For any fixed  $\varepsilon \in (0, \varepsilon_1)$ , there exists a nontrivial critical point  $(u_\varepsilon, v_\varepsilon)$  of  $J_\varepsilon$  such that  $u_\varepsilon \geq 0, v_\varepsilon \geq 0$  and  $J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon > 0$ . Moreover, at least one of  $u_\varepsilon > 0$  and  $v_\varepsilon > 0$  holds.*

**Proof.** Recall that  $\beta > \beta_0 \geq \max\{\mu_1, \mu_2\}$ . By (3.4) one has that

$$F_\varepsilon(x, s, t) \leq \gamma_\varepsilon(|x|) \sqrt{F(s, t)} \leq \frac{1}{2} \sqrt{\beta} \gamma_\varepsilon(|x|) (s^2 + t^2). \quad (3.18)$$

Then we deduce from  $(V_3)$ , (3.10) and (3.13) that

$$\begin{aligned} J_\varepsilon(u, v) &\geq \frac{1}{2} \|(u, v)\|_\varepsilon^2 - \int_{O_\varepsilon} F(u, v) dx - \frac{1}{2} \sqrt{\beta} \int_{\mathbb{R}^3 \setminus O_\varepsilon} \gamma_\varepsilon(|x|) (u^2 + v^2) dx \\ &\geq \frac{1}{4} \|(u, v)\|_\varepsilon^2 - C \int_{O_\varepsilon} (u^4 + v^4) dx \\ &\geq \frac{1}{4} \|(u, v)\|_\varepsilon^2 - C \|(u, v)\|_\varepsilon^4. \end{aligned}$$

Hence, there exists  $r, \alpha_1 > 0$  small such that

$$\inf_{\|(u, v)\|_\varepsilon = r} J_\varepsilon(u, v) = \alpha_1 > 0.$$

This implies that  $c_\varepsilon \geq \alpha_1 > 0$ . Choose  $\phi \in C_0^\infty(O_\varepsilon)$  such that  $\phi \geq 0$  and  $\phi \not\equiv 0$ . Then  $G_\varepsilon(x, \phi^+, \phi^+) = F(\phi, \phi)$ , which implies that  $J_\varepsilon(t\phi, t\phi) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . That is,  $J_\varepsilon$  has a mountain-pass structure. By Lemma 3.1 and Mountain Pass Theorem ([6]), there exists a nontrivial critical point  $(u_\varepsilon, v_\varepsilon)$  of  $J_\varepsilon$  such that  $J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon > 0$ . Denote  $u_\varepsilon^-(x) := \max\{-u_\varepsilon(x), 0\}$  and so is  $v_\varepsilon^-$ . Then we see from (3.12) that

$$\int_{\mathbb{R}^3} |\nabla u_\varepsilon^-|^2 + a_\varepsilon |u_\varepsilon^-|^2 dx = 0, \quad \int_{\mathbb{R}^3} |\nabla v_\varepsilon^-|^2 + b_\varepsilon |v_\varepsilon^-|^2 dx = 0,$$

which implies that  $u_\varepsilon, v_\varepsilon \geq 0$ . By the strong maximum principle, at least one of  $u_\varepsilon > 0$  and  $v_\varepsilon > 0$  holds. This completes the proof.  $\square$

Define

$$\tilde{c}_\varepsilon := \inf_{(u, v) \in \tilde{\Phi}_\varepsilon} \max_{t > 0} J_\varepsilon(tu, tv),$$

where  $\tilde{\Phi}_\varepsilon := \{(u, v) \in H_\varepsilon \setminus \{(0, 0)\} : \int_{O_\varepsilon} ((u^+)^2 + (v^+)^2) dx > 0\}$ . Then we have the following lemma.

**Lemma 3.3.** *For any  $(u, v) \in \tilde{\Phi}_\varepsilon$ , there exists a unique  $t_{u, v} > 0$  such that*

$$J_\varepsilon(t_{u, v}u, t_{u, v}v) := \max_{t > 0} J_\varepsilon(tu, tv). \quad (3.19)$$

Moreover,  $c_\varepsilon = \tilde{c}_\varepsilon$ .

**Proof.** Fix any  $(u, v) \in \tilde{\Phi}_\varepsilon$ . By the definition (3.6) of  $G_\varepsilon(x, u, v)$ , we have that for any  $x \in \mathbb{R}^3$ ,  $1/t \frac{d}{dt} G_\varepsilon(x, tu^+, tv^+)$  is nondecreasing as  $t > 0$  increases. Moreover, if  $x \in O_\varepsilon$  and  $(u^+)^2(x) + (v^+)^2(x) > 0$ , then  $1/t \frac{d}{dt} G_\varepsilon(x, tu^+, tv^+)$

is strictly increasing as  $t > 0$  increases. This means that there exists a unique  $t_{u,v} > 0$  such that  $\frac{d}{dt}J_\varepsilon(tu, tv)|_{t_{u,v}} = 0$ , that is,

$$\|u\|_{a,\varepsilon}^2 + \|v\|_{b,\varepsilon}^2 = \int_{\mathbb{R}^3} \frac{1}{t_{u,v}} \frac{d}{dt} G_\varepsilon(x, tu^+, tv^+)|_{t_{u,v}} dx.$$

Since

$$J_\varepsilon(tu, tv) \leq \frac{1}{2}t^2\|(u, v)\|_\varepsilon^2 - t^4 \int_{O_\varepsilon} F(u^+, v^+) dx \rightarrow -\infty \text{ as } t \rightarrow +\infty,$$

we see that (3.19) holds and  $c_\varepsilon \leq \max_{t>0} J_\varepsilon(tu, tv)$ . Therefore,  $c_\varepsilon \leq \tilde{c}_\varepsilon$ . Meanwhile, since  $(u_\varepsilon, v_\varepsilon) \in \tilde{\Phi}_\varepsilon$ ,  $c_\varepsilon = J_\varepsilon(u_\varepsilon, v_\varepsilon)$  and  $t_{u_\varepsilon, v_\varepsilon} = 1$ , we have  $c_\varepsilon \geq \tilde{c}_\varepsilon$ . This completes the proof.  $\square$

**Lemma 3.4.** *Let  $(u_\varepsilon, v_\varepsilon)$  be in Lemma 3.2. Then  $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq \inf_{P \in O} m(P)$ , and there exists  $C > 0$  independent of  $\varepsilon > 0$ , such that  $\|(u_\varepsilon, v_\varepsilon)\|_\varepsilon^2 \leq C$ .*

**Proof.** Fix any  $P \in O$  and let  $(U_P, V_P)$  be in Corollary 2.1. Take  $T > 0$  such that  $L_P(TU_P, TV_P) \leq -1$ . Note that there exists  $R > 0$  such that  $B(P, R) := \{x : |x - P| < R\} \subset O$ , we take  $\phi \in C_0^1(B(0, R), \mathbb{R})$  with  $0 \leq \phi \leq 1$  and  $\phi(x) \equiv 1$  for  $|x| \leq R/2$ . Define  $\phi_\varepsilon(x) := \phi(\varepsilon x)$ , then  $\phi_\varepsilon(x - P/\varepsilon) \neq 0$  implies  $x \in O_\varepsilon$ . Combining this with (3.6) and the Dominated Convergence Theorem, one has

$$\begin{aligned} & J_\varepsilon(t(\phi_\varepsilon U_P)(\cdot - P/\varepsilon), t(\phi_\varepsilon V_P)(\cdot - P/\varepsilon)) \\ &= \frac{t^2}{2} \int_{|x| \leq R/\varepsilon} (|\nabla(\phi_\varepsilon U_P)|^2 + a(\varepsilon x + P)\phi_\varepsilon^2 U_P^2) \\ & \quad + \frac{t^2}{2} \int_{|x| \leq R/\varepsilon} (|\nabla(\phi_\varepsilon V_P)|^2 + b(\varepsilon x + P)\phi_\varepsilon^2 V_P^2) - \int_{|x| \leq R/\varepsilon} F(t\phi_\varepsilon U_P, t\phi_\varepsilon V_P) \\ & \rightarrow L_P(tU_P, tV_P), \quad \text{as } \varepsilon \rightarrow 0, \text{ uniformly for } t \in [0, T]. \end{aligned}$$

So  $J_\varepsilon(T(\phi_\varepsilon U_P)(\cdot - P/\varepsilon), T(\phi_\varepsilon V_P)(\cdot - P/\varepsilon)) < 0$  for  $\varepsilon > 0$  sufficiently small. Since  $((\phi_\varepsilon U_P)(\cdot - P/\varepsilon), (\phi_\varepsilon V_P)(\cdot - P/\varepsilon)) \in \tilde{\Phi}_\varepsilon$ , we see from Lemma 3.3 that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} c_\varepsilon &\leq \limsup_{\varepsilon \rightarrow 0} \max_{t \in [0, T]} J_\varepsilon(t(\phi_\varepsilon U_P)(\cdot - P/\varepsilon), t(\phi_\varepsilon V_P)(\cdot - P/\varepsilon)) \\ &= \max_{t \in [0, T]} L_P(tU_P, tV_P) = L_P(U_P, V_P) = m(P). \end{aligned}$$

Since  $P \in O$  is arbitrary, we have  $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq \inf_{P \in O} m(P)$ . From (3.14), there exists  $C > 0$  independent of  $\varepsilon > 0$ , such that  $\|(u_\varepsilon, v_\varepsilon)\|_\varepsilon^2 \leq C$ . This completes the proof.  $\square$

**Lemma 3.5.** *Let  $(u_\varepsilon, v_\varepsilon)$  be in Lemma 3.2. Then there exists  $0 < \varepsilon_2 \leq \varepsilon_1$ , such that for any  $\varepsilon \in (0, \varepsilon_2)$ , there holds*

$$\|u_\varepsilon + v_\varepsilon\|_{L^\infty(O_\varepsilon)} \geq \sqrt{\min\{a_0, b_0\}/\beta}.$$

**Proof.** Without loss of generality, we assume that  $a_0 \leq b_0$ . Recall  $\alpha_0$  in (3.16), we take  $\varepsilon_2 \leq \varepsilon_1$  such that  $\frac{\alpha_0 \varepsilon_2^2}{\log \rho_0 / \varepsilon_2} \leq 1/4$ . Assume that  $\|u_\varepsilon + v_\varepsilon\|_{L^\infty(O_\varepsilon)} \leq \sqrt{a_0/\beta}$  for some  $\varepsilon \in (0, \varepsilon_2)$ , then

$$\mu_1 u_\varepsilon^3 + \beta u_\varepsilon v_\varepsilon^2 + \beta u_\varepsilon^2 v_\varepsilon + \mu_2 v_\varepsilon^3 \leq a_0(u_\varepsilon + v_\varepsilon) \text{ in } O_\varepsilon.$$

Combining this with (3.12) and (3.16) we obtain that

$$-\Delta(u_\varepsilon + v_\varepsilon) + a_\varepsilon u_\varepsilon + b_\varepsilon v_\varepsilon \leq \chi_{O_\varepsilon} a_0(u_0 + v_0) + \alpha_0(1 - \chi_{O_\varepsilon})\gamma_\varepsilon(|x|)(u_\varepsilon + v_\varepsilon),$$

that is,

$$-\Delta(u_\varepsilon + v_\varepsilon) \leq (1 - \chi_{O_\varepsilon}) \frac{\alpha_0 \varepsilon^2}{\log \rho_0 / \varepsilon} \frac{u_\varepsilon + v_\varepsilon}{|x|^2},$$

which implies from (3.11) that

$$\int_{\mathbb{R}^3} |\nabla(u_\varepsilon + v_\varepsilon)|^2 dx \leq \frac{\alpha_0 \varepsilon^2}{\log \rho_0 / \varepsilon} \int_{\mathbb{R}^3} \frac{|u_\varepsilon + v_\varepsilon|^2}{|x|^2} dx < \int_{\mathbb{R}^3} |\nabla(u_\varepsilon + v_\varepsilon)|^2 dx,$$

a contradiction. This completes the proof.  $\square$

**Lemma 3.6.** *Let  $(u_\varepsilon, v_\varepsilon)$  be in Lemma 3.2. Let  $(\varepsilon_n)_{n \geq 1}$  be a sequence with  $\varepsilon_n \rightarrow 0$ . Let  $k \geq 1$  and for  $i \in [1, k] \cap \mathbb{N}$ , there is  $\{P_n^i\}_{n \geq 1} \subset O_{\varepsilon_n}$  with  $\varepsilon_n P_n^i \rightarrow P^i \in \overline{O}$  as  $n \rightarrow \infty$ . If*

$$\liminf_{n \rightarrow \infty} (u_{\varepsilon_n} + v_{\varepsilon_n})(P_n^i) > 0, \quad \forall i; \quad \lim_{n \rightarrow \infty} |P_n^i - P_n^j| = +\infty, \quad \forall i \neq j.$$

*Then  $\liminf_{n \rightarrow \infty} c_{\varepsilon_n} \geq \sum_{i=1}^k m(P^i)$ .*

**Proof.** The proof is inspired by [28]. For  $i \in \{1, \dots, k\}$ , we define

$$(u_n^i, v_n^i) := (u_{\varepsilon_n}(\cdot + P_n^i), v_{\varepsilon_n}(\cdot + P_n^i)).$$

By Lemma 3.4,  $(u_{\varepsilon_n}, v_{\varepsilon_n})$  is uniformly bounded in  $H_\varepsilon$ , so  $u_n^i, v_n^i$  are uniformly bounded in  $H_{loc}^1(\mathbb{R}^3)$ . By the system and the elliptic regularity, it is standard to show that  $u_n^i, v_n^i$  are uniformly bounded in  $W_{loc}^{2,q}(\mathbb{R}^3)$  for any  $q \geq 2$ . By the compactness of Sobolev embedding, passing to a subsequence, we may assume that  $u_n^i \rightarrow u^i, v_n^i \rightarrow v^i$  strongly in  $C_{loc}^1(\mathbb{R}^3)$ . Moreover, we have  $u^i, v^i \geq 0$  and  $u^i(0) + v^i(0) > 0$ . By Fatou Lemma, for any  $R > 0$ , we have

$$\begin{aligned} \int_{B(0,R)} |\nabla u^i|^2 + a(P^i)|u^i|^2 dx &\leq \liminf_{n \rightarrow \infty} \int_{B(0,R)} |\nabla u_n^i|^2 + a(\varepsilon_n x + \varepsilon_n P_n^i)|u_n^i|^2 dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_{\varepsilon_n}|^2 + a_{\varepsilon_n}|u_{\varepsilon_n}|^2 dx \leq C. \end{aligned}$$

So  $u^i \in H^1(\mathbb{R}^3)$ . Similarly,  $v^i \in H^1(\mathbb{R}^3)$ . Since  $O$  is smooth, up to a subsequence, we may assume that  $\chi_{O_{\varepsilon_n}}(\cdot + P_n^i)$  converges almost everywhere to  $\chi^i$ ,

where  $0 \leq \chi^i \leq 1$ . In fact,  $\chi^i$  is either the characteristic function of  $\mathbb{R}^3$  or the characteristic function of a half space. Then it is easy to see that  $(u^i, v^i)$  satisfy

$$\begin{cases} -\Delta u + a(P^i)u = \chi^i(\mu_1 u^3 + \beta uv^2), & x \in \mathbb{R}^N, \\ -\Delta v + b(P^i)v = \chi^i(\mu_2 v^3 + \beta vu^2), & x \in \mathbb{R}^N, \\ u(x), v(x) \in H^1(\mathbb{R}^3). \end{cases} \quad (3.20)$$

Define

$$\begin{aligned} \tilde{L}_i(u, v) := & \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + a(P^i)u^2 + |\nabla v|^2 + b(P^i)v^2) dx \\ & - \frac{1}{4} \int_{\mathbb{R}^3} \chi^i(\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4) dx, \end{aligned} \quad (3.21)$$

then we see from (2.7) that

$$\tilde{L}_i(u^i, v^i) = \max_{t>0} \tilde{L}_i(tu^i, tv^i) \geq \max_{t>0} L_{P^i}(tu^i, tv^i) \geq m(P^i).$$

Define

$$H(n) := \frac{|\nabla u_{\varepsilon_n}|^2 + a_{\varepsilon_n}|u_{\varepsilon_n}|^2 + |\nabla v_{\varepsilon_n}|^2 + b_{\varepsilon_n}|v_{\varepsilon_n}|^2}{2} - G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n}). \quad (3.22)$$

Then,

$$\begin{aligned} & \liminf_{R \rightarrow \infty} \liminf_{n \rightarrow \infty} \int_{B(P_n^i, R)} H(n) dx \\ = & \liminf_{R \rightarrow \infty} \liminf_{n \rightarrow \infty} \int_{B(0, R)} \left( \frac{|\nabla u_n^i|^2 + a(\varepsilon_n x + \varepsilon_n P_n^i)|u_n^i|^2}{2} \right. \\ & \left. + \frac{|\nabla v_n^i|^2 + b(\varepsilon_n x + \varepsilon_n P_n^i)|v_n^i|^2}{2} - G_{\varepsilon_n}(x + P_n^i, u_n^i, v_n^i) \right) dx \\ = & \tilde{L}_i(u^i, v^i) \geq m(P^i). \end{aligned} \quad (3.23)$$

Similarly, we have

$$\limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{B(P_n^i, R) \setminus B(P_n^i, R/2)} (|\nabla u_{\varepsilon_n}|^2 + a_{\varepsilon_n}|u_{\varepsilon_n}|^2) dx = 0, \quad (3.24)$$

$$\limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{B(P_n^i, R) \setminus B(P_n^i, R/2)} (|\nabla v_{\varepsilon_n}|^2 + b_{\varepsilon_n}|v_{\varepsilon_n}|^2) dx = 0. \quad (3.25)$$

Define  $B_{R,n} := \mathbb{R}^3 \setminus \cup_{i=1}^k B(P_n^i, R)$ . We claim that

$$\liminf_{R \rightarrow \infty} \liminf_{n \rightarrow \infty} \int_{B_{R,n}} H(n) dx \geq 0. \quad (3.26)$$

From (3.8) and (3.9), one has

$$\int_{B_{R,n}} H(n) dx \geq \frac{1}{2} \int_{B_{R,n}} (|\nabla u_{\varepsilon_n}|^2 + a_{\varepsilon_n}|u_{\varepsilon_n}|^2 + |\nabla v_{\varepsilon_n}|^2 + b_{\varepsilon_n}|v_{\varepsilon_n}|^2)$$

$$-\nabla_{(u,v)} G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n})(u_{\varepsilon_n}, v_{\varepsilon_n}) dx := A_{R,n}. \quad (3.27)$$

Let  $\varphi_{R,n} \in C_0^\infty(\mathbb{R}^3)$  satisfy  $\varphi_{R,n} = 1$  on  $B_{R,n}$ ,  $\varphi_{R,n} = 0$  on  $\cup_{i=1}^k B(P_n^i, R/2)$  and  $|\nabla \varphi_{R,n}| \leq C/R$ . Recall that  $\nabla_{(u,v)} G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n})(u_{\varepsilon_n}, v_{\varepsilon_n}) \geq 0$ . Multiple the system (3.12) with  $(\varphi_{R,n} u_{\varepsilon_n}, \varphi_{R,n} v_{\varepsilon_n})$  and integrate over  $\mathbb{R}^3$ , we have

$$\begin{aligned} 2A_{R,n} &\geq - \int_{\cup_{i=1}^k B(P_n^i, R)} (|\nabla u_{\varepsilon_n}|^2 + a_{\varepsilon_n} |u_{\varepsilon_n}|^2 + |\nabla v_{\varepsilon_n}|^2 + b_{\varepsilon_n} |v_{\varepsilon_n}|^2) \varphi_{R,n} dx \\ &\quad - \int_{\cup_{i=1}^k B(P_n^i, R)} (u_{\varepsilon_n} \nabla u_{\varepsilon_n} \nabla \varphi_{R,n} + v_{\varepsilon_n} \nabla v_{\varepsilon_n} \nabla \varphi_{R,n}) := A_{R,n}^1 + A_{R,n}^2. \end{aligned}$$

From (3.24) and (3.25) we see that  $\liminf_{R \rightarrow \infty} \liminf_{n \rightarrow \infty} A_{R,n}^1 = 0$ . Since  $\cup_{i=1}^k B(P_n^i, R) \subset O_{\varepsilon_n}^{2\delta}$  for  $n$  large enough, we see from (3.2) and Lemma 3.4 that

$$\begin{aligned} &\limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\cup_{i=1}^k B(P_n^i, R)} |u_{\varepsilon_n} \nabla u_{\varepsilon_n} \nabla \varphi_{R,n}| dx \\ &\leq \limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} C/R \int_{\cup_{i=1}^k B(P_n^i, R)} |\nabla u_{\varepsilon_n}|^2 + a_{\varepsilon_n} |u_{\varepsilon_n}|^2 dx \\ &\leq \limsup_{R \rightarrow \infty} C/R = 0. \end{aligned}$$

Therefore,  $\liminf_{R \rightarrow \infty} \liminf_{n \rightarrow \infty} A_{R,n}^2 = 0$ . That is, (3.26) holds. By (3.23) and (3.26) we have

$$\liminf_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) \geq \sum_{i=1}^k \tilde{L}_i(u^i, v^i) \geq \sum_{i=1}^k m(P^i).$$

This completes the proof.  $\square$

**Lemma 3.7.** *Let  $(u_\varepsilon, v_\varepsilon)$  be in Lemma 3.2. Let  $P_\varepsilon \in O_\varepsilon$  such that  $\liminf_{\varepsilon \rightarrow 0} (u_\varepsilon + v_\varepsilon)(P_\varepsilon) > 0$ . Then*

$$\liminf_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon P_\varepsilon, \partial O) > 0, \quad \liminf_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \|u_\varepsilon + v_\varepsilon\|_{L^\infty(O_\varepsilon \setminus B(P_\varepsilon, R))} = 0.$$

**Proof.** Assume that there exists  $\varepsilon_n \rightarrow 0$  such that  $\lim_{n \rightarrow \infty} \text{dist}(\varepsilon_n P_{\varepsilon_n}, \partial O) = 0$ . Passing to a subsequence, we may assume that  $\varepsilon_n P_{\varepsilon_n} \rightarrow P_0 \in \partial O$ . By Lemmas 3.4 and 3.6 we have

$$\inf_{P \in O} m(P) \geq \lim_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) \geq m(P_0) \geq \inf_{P \in \partial O} m(P),$$

a contradiction with assumption  $(V_4)$ . Therefore,  $\liminf_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon P_\varepsilon, \partial O) > 0$ .

Assume that there exists  $y_n \in O_{\varepsilon_n}$  such that

$$|y_n - P_{\varepsilon_n}| \rightarrow +\infty, \quad \liminf_{n \rightarrow \infty} (u_{\varepsilon_n} + v_{\varepsilon_n})(y_n) > 0.$$



Passing to a subsequence, we may assume that  $\varepsilon_n y_n \rightarrow y_0 \in \overline{O}$  and  $\varepsilon_n P_{\varepsilon_n} \rightarrow P_0 \in O$ . Then by Lemma 3.6 again, we obtain

$$\inf_{P \in O} m(P) \geq \lim_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) \geq m(P_0) + m(y_0) \geq 2 \inf_{P \in O} m(P),$$

a contradiction. This completes the proof.  $\square$

By Lemmas 3.5 and 3.7, there exists  $x_\varepsilon \in O_\varepsilon$  such that

$$(u_\varepsilon + v_\varepsilon)(x_\varepsilon) = \max_{x \in O_\varepsilon} (u_\varepsilon + v_\varepsilon)(x) \geq \sqrt{\min\{a_0, b_0\}/\beta}. \quad (3.28)$$

The following lemma plays a crucial role in the proof of decay estimates.

**Lemma 3.8.** *Let  $(u_\varepsilon, v_\varepsilon)$  be in Lemma 3.2 and  $x_\varepsilon$  in (3.28). Let  $(\varepsilon_n)_{n \geq 1}$  be any a subsequence with  $\varepsilon_n \rightarrow 0$ . Then passing to a subsequence,  $\varepsilon_n x_{\varepsilon_n} \rightarrow P_0 \in \mathcal{M}$  and  $(u_{\varepsilon_n}(x + x_{\varepsilon_n}), v_{\varepsilon_n}(x + x_{\varepsilon_n}))$  converges to some  $(U, V) \in \mathcal{S}(P_0)$  strongly in  $C_{loc}^1(\mathbb{R}^3)$ , where  $\mathcal{S}(P_0)$  is defined in (2.3). Moreover,*

$$\lim_{n \rightarrow \infty} |\nabla u_{\varepsilon_n}(\cdot + x_{\varepsilon_n}) - \nabla U|_2 = 0, \quad \lim_{n \rightarrow \infty} |\nabla v_{\varepsilon_n}(\cdot + x_{\varepsilon_n}) - \nabla V|_2 = 0. \quad (3.29)$$

In particular, both  $u_\varepsilon > 0$  and  $v_\varepsilon > 0$  hold for  $\varepsilon > 0$  small enough.

**Proof.** By the proof of Lemma 3.7, it is easy to see that  $\varepsilon_n x_{\varepsilon_n} \rightarrow P_0 \in \mathcal{M}$ . Repeating the proof of Lemma 3.6, one has that  $(u_n(x), v_n(x)) := (u_{\varepsilon_n}(x + x_{\varepsilon_n}), v_{\varepsilon_n}(x + x_{\varepsilon_n}))$  converges to some  $(U, V) \in H$  strongly in  $C_{loc}^1(\mathbb{R}^3)$ . Since  $P_0 \in \mathcal{M} \cap O$ , we have that  $\chi_{O_{\varepsilon_n}}(\cdot + x_{\varepsilon_n})$  converges almost everywhere to 1. Therefore,  $(U, V)$  is a nontrivial solution of (1.4) with  $P = P_0$ . By Lemmas 3.4 and 3.6,

$$m(P_0) \geq \limsup_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) \geq \liminf_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) \geq L_{P_0}(U, V) \geq m(P_0).$$

Therefore,  $\lim_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) = L_{P_0}(U, V) = m(P_0)$ . By Corollary 2.1,  $U, V > 0$ . By (3.28),  $0 \in \mathbb{R}^3$  is a maximum point of  $U + V$ . Combining this with Remark 1.1, we see that  $U, V$  are radially symmetric, that is,  $(U, V) \in \mathcal{S}(P_0)$ .

We claim that

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{O_{\varepsilon_n} \setminus B(x_{\varepsilon_n}, R)} \nabla_{(u,v)} G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n})(u_{\varepsilon_n}, v_{\varepsilon_n}) dx = 0. \quad (3.30)$$

From (3.8), (3.9) and Fatou Lemma we have

$$\begin{aligned} L_{P_0}(U, V) &= \lim_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \frac{1}{2} \nabla_{(u,v)} G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n})(u_{\varepsilon_n}, v_{\varepsilon_n}) - G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n}) dx \\ &\geq \lim_{n \rightarrow \infty} \int_{O_{\varepsilon_n}} \frac{1}{2} \nabla_{(u,v)} G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n})(u_{\varepsilon_n}, v_{\varepsilon_n}) - G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n}) dx \end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \int_{O_{\varepsilon_n} \setminus B(x_{\varepsilon_n}, R)} F(u_{\varepsilon_n}, v_{\varepsilon_n}) dx + \lim_{n \rightarrow \infty} \int_{B(x_{\varepsilon_n}, R)} F(u_{\varepsilon_n}, v_{\varepsilon_n}) dx \\
&= \lim_{n \rightarrow \infty} \int_{(O_{\varepsilon_n} - x_{\varepsilon_n}) \setminus B(0, R)} F(u_n, v_n) dx + \lim_{n \rightarrow \infty} \int_{B(0, R)} F(u_n, v_n) dx \\
&\geq \int_{\mathbb{R}^3 \setminus B(0, R)} F(U, V) dx + \int_{B(0, R)} F(U, V) dx = L_{P_0}(U, V).
\end{aligned}$$

This implies that  $\lim_{n \rightarrow \infty} \int_{O_{\varepsilon_n} \setminus B(x_{\varepsilon_n}, R)} F(u_{\varepsilon_n}, v_{\varepsilon_n}) dx = \int_{\mathbb{R}^3 \setminus B(0, R)} F(U, V) dx$ . By (3.8) again, we see that (3.30) holds.

Next, we claim that

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B(x_{\varepsilon_n}, R)} |\nabla u_{\varepsilon_n}|^2 dx = 0. \quad (3.31)$$

Assume by contradiction that, up to a subsequence,

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B(x_{\varepsilon_n}, R)} |\nabla u_{\varepsilon_n}|^2 dx = 2\alpha_2 > 0. \quad (3.32)$$

Let  $H_n$  be in (3.22). Since  $\lim_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) = L_{P_0}(U, V)$ , by repeating the proof of Lemma 3.6 (especially see (3.23) and (3.26)), we deduce that

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B(x_{\varepsilon_n}, R)} H(n) dx = 0. \quad (3.33)$$

On the other hand, by (3.9) and Lemma 3.4 we have

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus O_{\varepsilon_n}} \nabla_{(u, v)} G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n})(u_{\varepsilon_n}, v_{\varepsilon_n}) dx \\
&\leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus O_{\varepsilon_n}} \sqrt{\beta} \gamma_{\varepsilon_n}(|x|)(u_{\varepsilon_n}^2 + v_{\varepsilon_n}^2) dx \\
&\leq \lim_{n \rightarrow \infty} \frac{\sqrt{\beta} \varepsilon_n^2}{\log \rho_0 / \varepsilon_n} \int_{\mathbb{R}^3 \setminus O_{\varepsilon_n}} \frac{u_{\varepsilon_n}^2 + v_{\varepsilon_n}^2}{|x|^2} dx \\
&\leq \lim_{n \rightarrow \infty} \frac{4\sqrt{\beta} \varepsilon_n^2}{\log \rho_0 / \varepsilon_n} \int_{\mathbb{R}^3} |\nabla u_{\varepsilon_n}|^2 + |\nabla v_{\varepsilon_n}|^2 dx = 0.
\end{aligned}$$

Combining this with (3.30), we get

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B(x_{\varepsilon_n}, R)} \nabla_{(u, v)} G_{\varepsilon_n}(x, u_{\varepsilon_n}, v_{\varepsilon_n})(u_{\varepsilon_n}, v_{\varepsilon_n}) dx = 0. \quad (3.34)$$

By (3.27), (3.32) and (3.34) we deduce that

$$\liminf_{R \rightarrow \infty} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B(x_{\varepsilon_n}, R)} H(n) dx \geq \alpha_2 > 0,$$

which contradicts with (3.33). Therefore, (3.31) holds, that is,

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B(0,R)} |\nabla u_n|^2 dx = 0.$$

Since  $u_n \rightarrow U$  strongly in  $C_{loc}^1(\mathbb{R}^3)$ , we have  $\lim_{n \rightarrow \infty} \int_{B(0,R)} |\nabla u_n - \nabla U|^2 dx = 0$  for any  $R > 0$ . Therefore,  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n - \nabla U|^2 dx = 0$ . Similarly,  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla v_n - \nabla V|^2 dx = 0$ , and so (3.29) holds. This means that both  $u_\varepsilon \not\equiv 0$  and  $v_\varepsilon \not\equiv 0$  for  $\varepsilon > 0$  small enough. By the strong maximum principle, we have  $u_\varepsilon, v_\varepsilon > 0$ . This completes the proof.  $\square$

**Lemma 3.9.** *Let  $(u_\varepsilon, v_\varepsilon)$  be in Lemma 3.2 and  $x_\varepsilon$  in (3.28). Then for  $\varepsilon > 0$  sufficiently small, there exists some  $c, C > 0$  independent of  $\varepsilon > 0$ , such that*

$$\omega_\varepsilon(x) := u_\varepsilon(x) + v_\varepsilon(x) \leq C \exp(-c \operatorname{dist}(x, \partial O_\varepsilon^{3\delta} \cup \{x_\varepsilon\})), \quad x \in O_\varepsilon^{3\delta}. \quad (3.35)$$

**Proof.** By (3.5) and (3.12) we have

$$\begin{cases} -\Delta u_\varepsilon + a_\varepsilon u_\varepsilon \leq \mu_1 u_\varepsilon^3 + \beta u_\varepsilon v_\varepsilon^2, & x \in \mathbb{R}^3, \\ -\Delta v_\varepsilon + b_\varepsilon v_\varepsilon \leq \mu_2 v_\varepsilon^3 + \beta u_\varepsilon^2 v_\varepsilon, & x \in \mathbb{R}^3. \end{cases}$$

Without loss of generality, we assume that  $a_0 \leq b_0$ . Then by (3.2) we get

$$-\Delta \omega_\varepsilon + \frac{a_0}{2} \omega_\varepsilon \leq \beta \omega_\varepsilon^3, \quad \text{in } O_\varepsilon^{5\delta}. \quad (3.36)$$

Then by elliptic estimates through the Moser iteration argument, we see that  $\{\|\omega_\varepsilon\|_{L^\infty(O_\varepsilon^{4\delta})}\}_\varepsilon$  is uniformly bounded. By the elliptic estimate [17, Theorem 8.17], there exists  $C > 0$  independent of small  $\varepsilon > 0$ , such that

$$\sup_{x \in B(y,1)} \omega_\varepsilon(x) \leq C \|\omega_\varepsilon\|_{L^2(B(y,2))} \leq C \|\omega_\varepsilon\|_{L^6(B(y,2))}, \quad \forall y \in O_\varepsilon^{3\delta}. \quad (3.37)$$

By (2.6) in Lemma 2.1, for any  $\sigma > 0$ , there exists  $R > 0$  large enough, such that

$$\|U\|_{L^6(\mathbb{R}^3 \setminus B(0,R))} \leq \sigma, \quad \|V\|_{L^6(\mathbb{R}^3 \setminus B(0,R))} \leq \sigma, \quad \forall (P, U, V) \in \mathcal{S}.$$

Note that  $6 = 2^*$  in dimension 3. By (3.29) in Lemma 3.8 we deduce that

$$\|u_\varepsilon\|_{L^6(\mathbb{R}^3 \setminus B(x_\varepsilon, R))} \leq 2\sigma, \quad \|v_\varepsilon\|_{L^6(\mathbb{R}^3 \setminus B(x_\varepsilon, R))} \leq 2\sigma, \quad \text{for } \varepsilon > 0 \text{ small enough.}$$

Combining this with (3.37), one has that

$$\sup_{y \in O_\varepsilon^{3\delta} \setminus B(x_\varepsilon, R+2)} \omega_\varepsilon(y) \leq c\sigma, \quad \text{for } \varepsilon > 0 \text{ small enough.}$$

By (3.36), there is a small  $\sigma > 0$  and so a large  $R > 0$ , such that

$$-\Delta \omega_\varepsilon + \frac{a_0}{4} \omega_\varepsilon \leq 0, \quad \text{in } O_\varepsilon^{3\delta} \setminus B(x_\varepsilon, R), \text{ for } \varepsilon > 0 \text{ small enough.}$$

Applying a comparison principle, there exists some  $C, c > 0$  independent of small  $\varepsilon > 0$ , such that (3.35) holds. This completes the proof.  $\square$

By  $(V_2)$  there exists  $R_1 > 0$  large enough, such that  $O \subset B(0, R_1)$  and for some  $c > 0$ ,

$$a(x), b(x) \geq \frac{c}{|x|^2 \log(|x|)}, \quad \forall |x| \geq R_1. \quad (3.38)$$

**Lemma 3.10.** *Let  $(u_\varepsilon, v_\varepsilon)$  be in Lemma 3.2 and  $x_\varepsilon$  in (3.28). Then for sufficiently large  $R_2 > R_1$ , there exists  $c, C > 0$  independent of  $\varepsilon > 0$ , such that*

$$\omega_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}}, \quad \text{for all } \delta/\varepsilon \leq |x - x_\varepsilon| \leq 2R_2/\varepsilon \quad (3.39)$$

holds for  $\varepsilon > 0$  sufficiently small.

**Proof.** Let  $D := \{x \in \mathbb{R}^3 : a(x) = 0 \text{ or } b(x) = 0\}$ . Then by  $(V_2) - (V_3)$  we see that  $D \subset \mathbb{R}^3 \setminus O$  is compact. First, we claim that, for any sufficiently large  $R_2 > R_1$  and sufficiently small  $l > 0$ , there exists  $C, c > 0$  independent of small  $\varepsilon > 0$ , such that

$$\omega_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}}, \quad \text{for } \delta/\varepsilon \leq |x - x_\varepsilon| \leq 2R_2/\varepsilon \text{ and } \text{dist}(\varepsilon x, D) \geq l. \quad (3.40)$$

By the definition of  $D$ , we may assume that

$$\inf_{x \in B(x_\varepsilon, 4R_2/\varepsilon) \setminus D_\varepsilon^{l/4}} \min\{a_\varepsilon(x), b_\varepsilon(x)\} = a' > 0.$$

By a similar proof of Lemma 3.9, we can get that

$$-\Delta \omega_\varepsilon + \frac{a'}{2} \omega_\varepsilon \leq 0, \quad \delta/\varepsilon \leq |x - x_\varepsilon| \leq 2R_2/\varepsilon \text{ and } \text{dist}(\varepsilon x, D) \geq l/2$$

holds for  $\varepsilon > 0$  small enough. Applying a comparison principle again, there exists some  $C, c > 0$  independent of small  $\varepsilon > 0$ , such that

$$\omega_\varepsilon(x) \leq C \exp(-c \text{dist}(x, \partial D_\varepsilon^{l/2} \cup \{x_\varepsilon\})) \quad (3.41)$$

holds for  $\delta/\varepsilon \leq |x - x_\varepsilon| \leq 2R_2/\varepsilon$  and  $\text{dist}(\varepsilon x, D) \geq l/2$ . That is, (3.40) holds.

Let  $l > 0$  small enough such that  $D^{2l} \cap O = \emptyset$ . Let  $\psi \geq 0$  satisfy

$$\begin{cases} -\Delta \psi = \lambda_1 \psi, & x \in D^{2l}, \\ \psi = 0, & x \in \partial D^{2l}, \end{cases}$$

where  $\lambda_1$  is the first eigenvalue. We may assume that  $\max_{x \in D^{2l}} \psi(x) = 1$ . Define  $\psi_\varepsilon(x) := \psi(\varepsilon x)$ . By (3.16) we see that

$$\begin{aligned} & -\Delta \psi_\varepsilon + a_\varepsilon \psi_\varepsilon - \frac{\partial_u G_\varepsilon(x, u_\varepsilon, v_\varepsilon)}{u_\varepsilon} \psi_\varepsilon \geq \lambda_1 \varepsilon^2 \psi_\varepsilon - \alpha_0 \gamma_\varepsilon(|x|) \psi_\varepsilon \\ & \geq \varepsilon^2 \left( \lambda_1 - \frac{\alpha_0}{|\rho_0/\varepsilon|^2 \log |\rho_0/\varepsilon|} \right) \psi_\varepsilon \geq 0 \quad \text{in } D_\varepsilon^{2l}, \text{ for } \varepsilon > 0 \text{ small enough.} \end{aligned}$$

Therefore, by (3.40) and a comparison principle, there exists some  $C, c > 0$  independent of small  $\varepsilon > 0$ , such that

$$u_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}} \psi_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}} \text{ for } x \in D_\varepsilon^l.$$

Similarly, we can prove that  $v_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}}$  for  $x \in D_\varepsilon^l$ . Therefore,  $\omega_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}}$  for  $x \in D_\varepsilon^l$ . Combining this with (3.40), we see that (3.39) holds. This completes the proof.  $\square$

**Lemma 3.11.** *Let  $(u_\varepsilon, v_\varepsilon)$  be in Lemma 3.2,  $x_\varepsilon$  in (3.28) and  $R_2$  in Lemma 3.10. Then for any  $\alpha > 0$ , there exists  $c, C > 0$  independent of  $\varepsilon > 0$ , such that*

$$\omega_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}} |x|^{-1} |\log |x||^{-\alpha}, \quad \text{for all } x \in \mathbb{R}^3 \setminus B(0, R_2/\varepsilon) \quad (3.42)$$

*holds for  $\varepsilon > 0$  sufficiently small. In particular, there exists  $\varepsilon_0 > 0$  small enough, such that for any  $\varepsilon \in (0, \varepsilon_0)$ ,  $(u_\varepsilon, v_\varepsilon)$  is a positive vector solution of (3.1).*

**Proof.** The following proof is similar to [7], and we give the proof for the completeness. For any fixed  $\alpha > 0$ , we define

$$\Gamma_\varepsilon(x) = \frac{1}{|x|(\log |x|)^\alpha}, \quad (3.43)$$

then there exists some  $C > 0$  such that  $\min_{x \in \partial O_\varepsilon} \Gamma_\varepsilon(x) \geq C\varepsilon^2$ . For any  $x \in \mathbb{R}^3 \setminus B(0, R_2/\varepsilon)$ , we have

$$\Delta \Gamma_\varepsilon(x) = \frac{\alpha}{|x|^3 (\log |x|)^{\alpha+1}} + \frac{\alpha(\alpha+1)}{|x|^3 (\log |x|)^{\alpha+2}},$$

and so it follows from (3.16) and (3.38) that

$$\begin{aligned} & \left( -\Delta \Gamma_\varepsilon + a_\varepsilon \Gamma_\varepsilon - \frac{\partial_u G_\varepsilon(x, u_\varepsilon, v_\varepsilon)}{u_\varepsilon} \Gamma_\varepsilon \right) / \Gamma_\varepsilon \\ & \geq \frac{c}{|\varepsilon x|^2 \log |\varepsilon x|} - \frac{\alpha}{|x|^2 \log |x|} - \frac{\alpha(\alpha+1)}{|x|^2 (\log |x|)^2} - \frac{\alpha_0 \varepsilon^2}{|x|^2 \log |x|}. \end{aligned}$$

Since for small  $\varepsilon > 0$  and  $|x| \geq R_2/\varepsilon$ ,

$$\frac{1}{|\varepsilon x|^2 \log |\varepsilon x|} \geq \frac{1}{\varepsilon(\varepsilon+1)} \frac{1}{|x|^2 \log |x|},$$

we see that

$$-\Delta \Gamma_\varepsilon + a_\varepsilon \Gamma_\varepsilon - \frac{\partial_u G_\varepsilon(x, u_\varepsilon, v_\varepsilon)}{u_\varepsilon} \Gamma_\varepsilon \geq 0 \text{ in } \mathbb{R}^3 \setminus B(0, R_2/\varepsilon).$$

For  $x \in \partial B(0, R_2/\varepsilon)$ , since  $\varepsilon x \notin O$  and  $\varepsilon x_\varepsilon \in \mathcal{M}^\delta$  for  $\varepsilon > 0$  sufficiently small by Lemma 3.8, we have

$$2R_2/\varepsilon \geq |x - x_\varepsilon| \geq \frac{|\varepsilon x - \varepsilon x_\varepsilon|}{\varepsilon} \geq \frac{\text{dist}(\mathbb{R}^3 \setminus O, \mathcal{M}^\delta)}{\varepsilon} \geq \frac{4\delta}{\varepsilon},$$

by (3.39) in Lemma 3.10,  $u_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}}$  for all  $x \in \partial B(0, R_2/\varepsilon)$ . Therefore, by a comparison principle, there exists  $C, c > 0$  independent of small  $\varepsilon > 0$  such that

$$u_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}}|x|^{-1}|\log|x||^{-\alpha}, \quad \text{for all } x \in \mathbb{R}^3 \setminus B(0, R_2/\varepsilon)$$

holds for  $\varepsilon > 0$  sufficiently small. By a similar proof, the conclusion also holds for  $v_\varepsilon$ . That is, (3.42) holds for  $\varepsilon > 0$  sufficiently small.

Now we fix a  $\alpha > 1/2$ . For any  $x \in \mathbb{R}^3 \setminus B(0, R_2/\varepsilon)$ , we have

$$\begin{aligned} 4F(u_\varepsilon(x), v_\varepsilon(x)) &= \mu_1 u_\varepsilon^4(x) + 2\beta u_\varepsilon^2(x) v_\varepsilon^2(x) + \mu_2 v_\varepsilon^4(x) \\ &\leq \beta \omega_\varepsilon^4(x) \leq Ce^{-\frac{4c}{\varepsilon}}|x|^{-4}|\log|x||^{-4\alpha} \\ &< \frac{\varepsilon^4}{|x|^4(\log|x|)^2} = \gamma_\varepsilon(|x|)^2 \quad \text{for } \varepsilon > 0 \text{ small enough.} \end{aligned}$$

For  $x \in B(0, R_2/\varepsilon) \setminus O_\varepsilon$ , we have  $\delta/\varepsilon < \rho_0/\varepsilon \leq |x| \leq R_2/\varepsilon$ . Then by (3.39) in Lemma 3.10, we deduce that

$$4F(u_\varepsilon(x), v_\varepsilon(x)) \leq Ce^{-\frac{4c}{\varepsilon}} < \frac{\varepsilon^4}{|x|^4(\log|x|)^2} = \gamma_\varepsilon(|x|)^2 \quad \text{for } \varepsilon > 0 \text{ small enough.}$$

Therefore, there exists  $\varepsilon_0 > 0$  sufficiently small, such that for any  $\varepsilon \in (0, \varepsilon_0)$ , we have

$$F(u_\varepsilon(x), v_\varepsilon(x)) < \frac{1}{4}\gamma_\varepsilon(|x|)^2, \quad \forall x \in \mathbb{R}^3 \setminus O_\varepsilon,$$

which implies that  $G_\varepsilon(x, u_\varepsilon, v_\varepsilon) \equiv F(u_\varepsilon, v_\varepsilon)$ , and so  $(u_\varepsilon, v_\varepsilon)$  is a positive vector solution of (3.1). This completes the proof.  $\square$

**Completion of the proof of Theorem 1.1.** Let  $\varepsilon \in (0, \varepsilon_0)$ . Since  $x_\varepsilon \in O_\varepsilon$ , we have  $|x_\varepsilon| < R_2/\varepsilon$  and so

$$|x| > R_2/\varepsilon, \quad |x - x_\varepsilon| \leq 2|x|, \quad \forall x \in \mathbb{R}^3 \setminus B(x_\varepsilon, 2R_2/\varepsilon).$$

Combining these with (3.42), there exists  $C, c > 0$  independent of  $\varepsilon$  such that

$$\omega_\varepsilon(x) \leq Ce^{-\frac{c}{\varepsilon}}|x - x_\varepsilon|^{-1}|\log|x - x_\varepsilon||^{-\alpha}, \quad \text{for all } x \in \mathbb{R}^3 \setminus B(x_\varepsilon, 2R_2/\varepsilon). \quad (3.44)$$

Define  $(\tilde{u}_\varepsilon(x), \tilde{v}_\varepsilon(x)) := (u_\varepsilon(x/\varepsilon), v_\varepsilon(x/\varepsilon))$  and  $\tilde{x}_\varepsilon := \varepsilon x_\varepsilon$ . Then  $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$  is a positive vector solution of (1.3). Moreover,  $\tilde{x}_\varepsilon$  is a maximum point of  $\tilde{u}_\varepsilon + \tilde{v}_\varepsilon$ . Conclusions (i) and (ii) in Theorem 1.1 follow directly from Lemma 3.8. By (3.44) we have that

$$\begin{aligned} \tilde{u}_\varepsilon(x) + \tilde{v}_\varepsilon(x) &\leq Ce^{-\frac{c}{\varepsilon}}|x/\varepsilon - x_\varepsilon|^{-1}|\log|x/\varepsilon - x_\varepsilon||^{-\alpha} \\ &= Ce^{-\frac{c}{\varepsilon}} \frac{\varepsilon}{|x - \tilde{x}_\varepsilon||\log(|x - \tilde{x}_\varepsilon|/\varepsilon)|^\alpha} \\ &\leq Ce^{-\frac{c}{\varepsilon}} \frac{1}{|x - \tilde{x}_\varepsilon||\log(|x - \tilde{x}_\varepsilon| + 2)|^\alpha} \end{aligned} \quad (3.45)$$

holds for all  $x \in \mathbb{R}^3 \setminus B(\tilde{x}_\varepsilon, 2R_2)$ . By (3.39) in Lemma 3.10, there exists some  $C, c > 0$  independent of  $\varepsilon \in (0, \varepsilon_0)$ , such that (3.45) holds for all  $x \in \mathbb{R}^3 \setminus B(\tilde{x}_\varepsilon, \delta)$ .

For any  $x \in B(\tilde{x}_\varepsilon, \delta)$ , since  $\tilde{x}_\varepsilon = \varepsilon x_\varepsilon \in \mathcal{M}^\delta$  for  $\varepsilon > 0$  small, we have  $\text{dist}(x, \partial O^{3\delta} \cup \{\tilde{x}_\varepsilon\}) = |x - \tilde{x}_\varepsilon|$ . By (3.35) in Lemma 3.9, we get that

$$\tilde{u}_\varepsilon(x) + \tilde{v}_\varepsilon(x) \leq C \exp\left(-\frac{c}{\varepsilon}|x - \tilde{x}_\varepsilon|\right), \quad x \in B(\tilde{x}_\varepsilon, \delta).$$

Therefore, (iii) in Theorem 1.1 holds. This completes the proof.  $\square$

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